The birth of string theory
or
How to build a theory without a Lagrangian

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Foreword

Some material about the origin of string theory can be found in

P. Di Vecchia, Lect. Notes in Phys. **737** (2008) 59, 0704.0101 [hep-th]

and

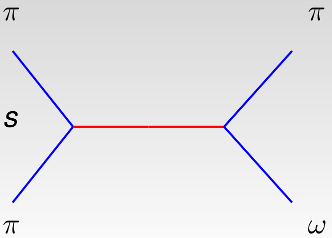
A. Cappelli, E. Castellani, F. Colomo and P. Di Vecchia, The Birth of String Theory, Cambridge University press 2012.

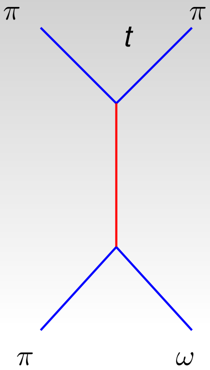
Plan of the talk

The dual resonance model

- After the successful formulation of QED during the forties it was natural to try to apply QFT to strong interactions.
- Because of the large number of hadrons and the strength of the pion-nucleon coupling constant it did not seem useful to write a Lagrangian describing the hadrons and their interaction.
- Under the influence from the Berkeley and Cambridge schools the trend was to forget not directly observable quantities as the Lagrangian and to try to write directly the observable ones, as the S-matrix, imposing the properties that it was supposed to satisfy.
- Those properties are analyticity, crossing symmetry, unitarity and Regge behaviour. Use bootstrap to impose them.
- Two important books on this topics: [Chew, The analytic S matrix, 1966; Eden, Landshoff, Olive and Polkinghorne, The analytic S-matrix, 1966]
- The most important result that came out from S-matrix theory has been the Veneziano model [Veneziano, 1968].
- How was it found?

Scattering amplitude for $\pi\pi \to \pi\omega$(spin 1)





- Amplitude depends on the Mandelstam variables

$$s = -(p_1 + p_2)^2 \;\; ; \;\; t = -(p_1 + p_4)^2 \;\; ; \;\; u = -(p_1 + p_3)^2$$

$$s + t + u = \sum_{i=1} m_i^2$$

- Exchange of resonances R in the s-channel at low energy

$$A_{\pi\pi\to\pi\omega} = \frac{2M_R A_{\pi\pi R} A_{R\pi\omega}}{s_R - s - iM_R\Gamma} \quad ; \quad \Gamma = \sum_i \Gamma_i$$

  Unitarity imposes that the sum of the partial widths be equal to the total width Γ. At the pole the amplitude factorises.
- Exchange of Regge poles in the t-channel at high energy (s large, t negative and small)

$$A(s,t) \sim \frac{\pi\beta(t)}{\Gamma(\alpha(t))\sin\pi\alpha(t)} \left(\frac{s}{s_0}\right)^{\alpha(t)} e^{i\pi\alpha(t)}$$

- $\alpha(t) = \alpha_0 + \alpha' t + \dots$ is the Regge trajectory and $\beta(t)$ is the residue of the Regge pole. The Regge residue also factorises.
- They are fixed from experiments as the parameters of the resonances.
- The Regge amplitude has a pole when $\alpha(t)$ is equal to a non negative integer n.
- The entire Regge trajectory is exchanged, not just a single pole as in the case of a resonance at low energy.

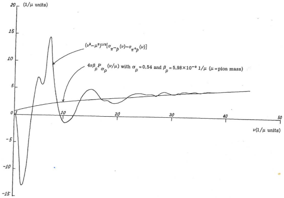


FIG. 1. The integrands of the sum rule (9) are plotted: The values of $(\nu^2-\mu^2)^{1/2}[\sigma_{\pi^-p}(\nu)-\sigma_{\pi^+p}(\nu)]$ are taken from the experiments, and the values of $4\pi\beta_\rho P_{\alpha_\rho}(\nu/\mu)$ are calculated with $\alpha_\rho = 0.54$ and $\beta_\rho = 5.98\times10^{-2}\mu^{-1}$.

From Igi and Matsuda, Phys. Rev. Letters,18 (1967) 625

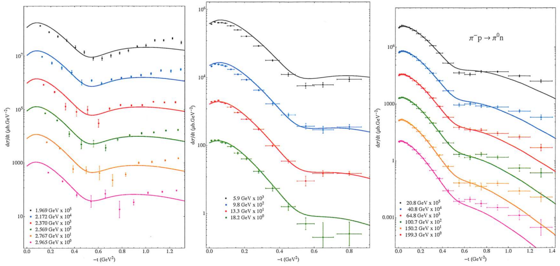


FIG. 3. (color online) $\pi^- p \to \pi^0 n$ differential cross sections from $p_{\rm lab}$=1.969 GeV to $p_{\rm lab}$=199.3 GeV. Scaling factors are indicated on the figure. The theoretical model includes the ρ pole (solid line). Data from [23–25].

proximated by the ρ Regge-pole and are given by

$$A'^{(-)} = \pi C_0^\rho \frac{\left[(1+C_2^\rho)e^{C_1^\rho t} - C_2^\rho\right]}{\Gamma(\alpha_\rho + 1)} \frac{e^{-i\pi\alpha_\rho} - 1}{2\sin\pi\alpha_\rho} \nu^{\alpha_\rho}, \tag{21a}$$

$$B^{(-)} = -D_0^\rho e^{D_1^\rho t} \frac{\pi}{\Gamma(\alpha_\rho)} \frac{e^{-i\pi\alpha_\rho} - 1}{2\sin\pi\alpha_\rho} \nu^{\alpha_\rho - 1}. \tag{21b}$$

The energy dependence is chosen such that the differential cross section behaves as $d\sigma/dt \sim s^{2\alpha_\rho - 2}$ at large energies. The relative sign is such that the imaginary part of $A'^{(-)}$ and $B^{(-)}$ have the same sign as C_0^ρ and D_0^ρ, respectively.

In the following we use a linear trajectory for the ρ pole, $\alpha_\rho = \alpha_\rho^0 + \alpha'_\rho t$. We first determine the parameters of the ρ trajectories using only the data on the charge exchange reaction $\pi^- p \to \pi^0 n$. Since the parameter C_2^ρ is sensitive to the cross-over between $\pi^- p$ and $\pi^+ p$ elastic scattering, our first fit cannot be used to determine C_2^ρ. We then impose the relation $C_2^\rho = \left[e^{0.1 C_1^\rho} - 1\right]^{-1}$ such that the cross over arises at ... 1 GeV^2.
At this stage, our model for the t-channel ρ exchange involves six parameters: magnitudes of the two residues,

TABLE I. Regge pole parameters.

x	ρ	$\mathbb{P}$	f
α_x^0	0.490 ± 0.003	1.075 ± 0.001	0.490
α'_x	0.943 ± 0.009	0.434 ± 0.002	0.943
α''_x	–	0.162 ± 0.007	–
C_0^x	5.01 ± 0.09	23.89 ± 0.09	71.35 ± 0.29
C_1^x	10.10 ± 0.21	2.21 ± 0.02	3.18 ± 0.04
D_0^x	128.87 ± 2.86	–	–
D_1^x	1.38 ± 0.07	–	–

neglect other contributions like Regge cuts and the ρ' daughter trajectory. One can therefore assume a power law behavior for the energy dependence of the differential cross section and extract the ρ trajectory from

$$\alpha_{\rm eff}(t) = \frac{1}{2}\log\left(\frac{p_a^2 d\sigma(p_a,t)/dt}{p_b^2 d\sigma(p_b,t)/dt}\right)\log^{-1}\left(\frac{\nu_a}{\nu_b}\right). \tag{22}$$

We compare the effective trajectory extracted from the data ... using p_a = 150.2 GeV and p_b = 199.2 GeV in Eq. (22) and from our model in Fig. 4. They clearly

From Mathieu et al, 1506.01764 [hep-ph]

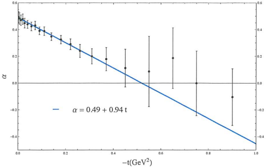


FIG. 4. (color online) ρ trajectories from our model (blue solid line) and Barger and Phillips [?] (green dashed line) compared to effective trajectory extracted from data with Eq. (22). We use data at $p_{\rm lab} = 20.8$ and 199.3 GeV from [25].

The low-energy contribution to the FESR in Fig. 1 indicates that helicity-flip $\nu B^{(+)}$ and helicity non-flip $A'^{(+)}$ isoscalar $t-$channel amplitudes are comparable. Phenomenologically the helicity non-flip amplitude $A'^{(+)}$, proportional to the total cross section, is more constrained by the data than the helicity flip amplitude $\nu B^{(+)}$. We choose to impose the equality between $t-$channel helicity flip and non-flip amplitudes in order to satisfy the FESR. The first physical particle on the f_2 trajectory is the $f_2(1275)$ spin-2 meson. To remove the ghost pole at $\alpha_f = 0$ we use the parametrization

$$A'^{\mathbb{P}} = -C_0^{\mathbb{P}} e^{C_1^{\mathbb{P}} t} \frac{\pi}{\Gamma(\alpha_{\mathbb{P}})} \frac{e^{-i\pi\alpha_{\mathbb{P}}}+1}{2\sin\pi\alpha_{\mathbb{P}}} \nu^{\alpha_{\mathbb{P}}}, \quad \nu B^{\mathbb{P}} = A'^{\mathbb{P}}, \tag{24a}$$

$$A'^{f} = -C_0^{f} e^{C_1^{f} t} \frac{\pi}{\Gamma(\alpha_f)} \frac{e^{-i\pi\alpha_f}+1}{2\sin\pi\alpha_f} \nu^{\alpha_f}, \quad \nu B^{f} = A'^{f}. \tag{24b}$$

From Mathieu et al, 1506.01764 [hep-ph]

We choose the f_2 trajectory to be degenerate with the ρ, $\alpha_f = \alpha_\rho$. The degeneracy between the ρ and f_2 trajectories and residues follows from absence of exotic, isospin-2 mesons, *e.g.* in $\pi^+\pi^+$ scattering [29]. Degeneracy between the f_2 and ρ and absence of ghost poles ($\alpha_f = 0$) is then consistent with the observed zero in the ρ residue at $\alpha_\rho = 0$ *cf.* Eq. (21b).

The Pomeron trajectory has a special status. There are no known mesons lying on it, with the exception that it may be related to the tensor glueball [30]. The trajectory is known to be approximately constant, $\alpha^{\mathbb{P}} \sim 1$. In the following we parametrize it using a second order polynomial,

$$\alpha_{\mathbb{P}} = \alpha_{\mathbb{P}}^0 + \alpha'_{\mathbb{P}} t + \alpha''_{\mathbb{P}} t^2, \tag{25}$$

constant, and whether or not the factor $\Gamma(\alpha^{\mathbb{P}})$ is included is a matter of taste.

In total we thus have seven parameters describing the leading t-channel isoscalar Regge poles. Initially we attempted to fix these parameters, just like we did in the case of isovector exchanges, by fitting the differential cross section. Since the Pomeron exchange, having the largest intercept, dominates and at the same time has a weak t-dependent, we found that the error on the magnitude of the residue was large, of the order of 10%. We therefore chose to perform a fit of the total cross sections (keeping only $p_{\rm lab} \geq 5$ GeV data) to first determine $C_0^{\mathbb{P},f}$ and $\alpha_{\mathbb{P}}^0$ for the Pomeron. The results are shown in Fig. 5. In the next step, using the differential cross section for $p_{\rm lab} > 3$ GeV we determine the f_2 and Pomeron slope parameters $C_1^{\mathbb{P},f}$, and the remaining Pomeron parameters that determine its t-dependence, $\alpha'_{\mathbb{P}}$ and $\alpha''_{\mathbb{P}}$. The comparison with the data is shown in Fig. 6 for $p_{\rm lab} \geq 50$ GeV. In the fit we use the data from [31-33]. The value of the parameters is given in columns three and four in Table I.

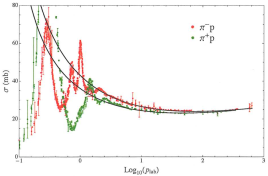


FIG. 5. (color online) Total cross section. Data from [6].

We compare our model with the differential cross section at $p_{\rm lab} = 3, 5, 6$ GeV from Ref. [33] as shown on Fig. 7. Our amplitudes reproduce the $\pi^\pm p$ differential cross section in whole range of t.

In the model the isovector contributions to the helicity non-flip amplitude is almost negligible. If follows from Eq. (10c), that with the approximation $A'^{(-)} \approx 0$ polarizations in $\pi^+ p$ and $\pi^- p$ elastic scattering are predicted to be opposite to each other. This is verified at energies higher than $p_{\rm lab} > 5$ GeV, *cf.* as shown in Fig 8.

C. Comparison between low- and high-energy contributions to the sum rules

- An important tool used to constrain the scattering amplitude was the use of Finite Energy Sum Rules (derived from Cauchy theorem + analyticity + asymptotic behaviour)

$$\int_0^{\bar{\nu}} \nu^n ImA(\nu, t) d\nu = \frac{\beta(t)}{\alpha(t)+n} \left(\frac{\bar{\nu}}{\nu_1}\right)^{\alpha(t)-1} \bar{\nu}^{n+1} \quad ; \quad \nu = \frac{1}{4}(s-u)$$

by saturating the low energy part of the amplitude with resonances obtaining the high energy part with Regge poles. [Igi and Matsuda, 1967; Dolan, Horn, Schmidt, 1967 and many others]

- This method was used for the process $\pi\pi \to \pi\omega$ obtaining

$$ImA(s,t) = \frac{\pi\beta(t)}{\Gamma(\alpha(t))} (\alpha' s)^{\alpha(t)-1} \left(1 + \mathcal{O}(1/s)\right)$$

with $\beta(t) = constant, \alpha(t) = \alpha_0 + \alpha' t$
[Ademollo, Rubinstein, Veneziano and Virasoro, 1968].

- Arriving at the amplitude

$$A(s,t) = \beta(t)\Gamma(1-\alpha(t))(-\alpha' s)^{\alpha(t)-1} \left(1 + \mathcal{O}(1/s)\right)$$

- The amplitude that provides the previous Regge behaviour and is crossing symmetric if we exchange s with t is:

$$A(s,t) = \beta\frac{\Gamma(1-\alpha(t))\Gamma(1-\alpha(s))}{\Gamma(2-\alpha(s)-\alpha(t))} = \beta B(1-\alpha(t);1-\alpha(s))$$

  where $B(1-\alpha(t);1-\alpha(s))$ is the Euler Beta-function [Veneziano, 1968, rec. 29 July 1968].
- The full $\pi\pi \to \pi\omega$ scattering amplitude is given by:

$$A(\pi\pi \to \pi\omega) = \beta\Big(B(1-\alpha(t);1-\alpha(s)) + B(1-\alpha(t);1-\alpha(u)) + B(1-\alpha(s);1-\alpha(u))\Big)$$

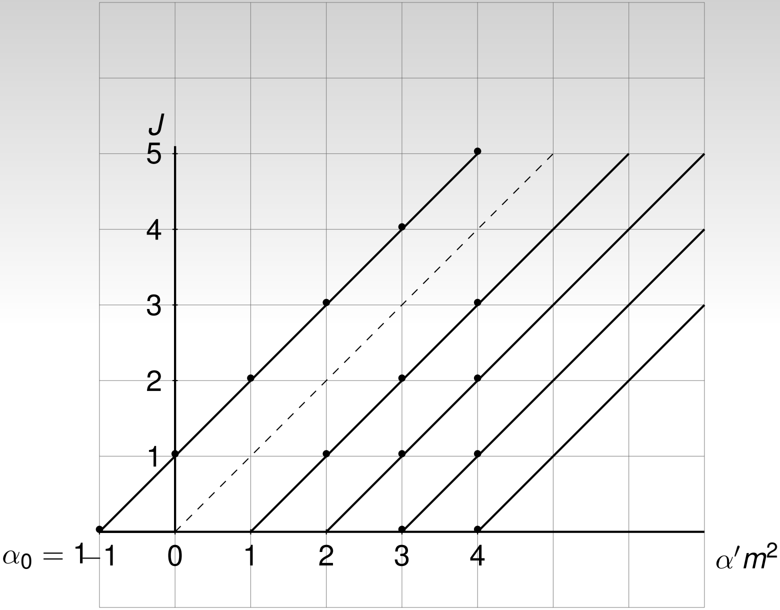
- It is completely symmetric under the exchange of s, t, u.

- The previous amplitude was immediately generalised to the elastic scattering of two scalar particles (not pions, for pions see below)

$$A(s,t) \sim \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))} \;\; ; \;\; s = -(p_1+p_2)^2 \;\; ; \;\; t = -(p_1-p_4)^2$$

  with linearly rising Regge trajectories: $\alpha(s) = \alpha_0 + \alpha' s$.
- For $\alpha(s_n) = n \geq 0$ the Γ-function has a pole that corresponds to a resonance with $M^2 = s_n$.
- From the residue of the pole one can read the spin of the particle exchanged.
- It turned out that, at the level n, we get particles with spin from 0 to n.
- The model contains an infinite number of resonances with zero width.
- At high energy($s \to \infty$, t fixed) the amplitude shows Regge behavior:

$$A(s,t) \Longrightarrow (-\alpha(s))^{\alpha(t)}\Gamma(\alpha(t))$$


J
5
4
3
2
1
$\alpha_0 = 1$
−1
0
1
2
3
4
$\alpha' m^2$

- Soon after, it was generalised to N scalar particles.
- But in 1968 there was no Lagrangian and actually the idea was to use the principles of S-matrix theory to construct directly the observables as the S-matrix.
- The N-point amplitude was constructed according to two principles.
- The first was the requirement to have only an infinite number of poles in all $\frac{N(N-3)}{2}$ planar channels.
- The second one was to exclude simultaneous poles in incompatible channels.
- This is a consequence of writing the amplitude in terms of the duality diagrams where the two lines describe the quark and the antiquark making the mesons but, at that time, NOT the string joining the two [Harari,1969] and [Rosner, 1969]
- In the case of the four-point amplitude one has two 2-particle channels (s and t).
- One introduces an integration variable for each channel X_s and X_t.
- They are incompatible because we cannot have simultaneous poles in those channels.

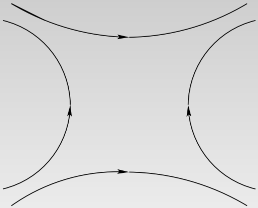


Figure: Duality diagram for the scattering of four mesons

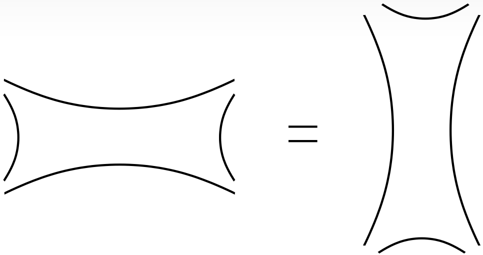


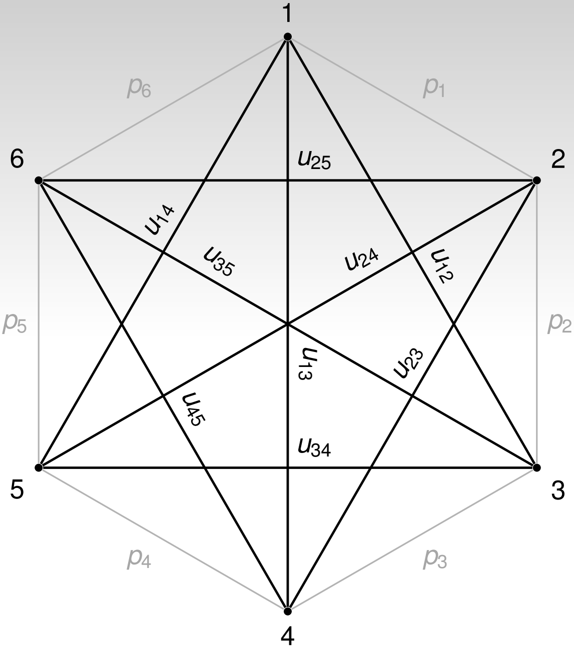
Figure: The duality diagram contains both s and t channel poles, but not simultaneously

- One then writes the amplitude as follows:

$$A_4 = \int_0^1 dX_s X_s^{-\alpha(s)-1} \int_0^1 dX_t X_t^{-\alpha(t)-1} \delta(X_s + X_t - 1)$$
$$= \int_0^1 dX_s X_s^{-\alpha(s)-1} (1 - X_s)^{-\alpha(t)-1}$$

where the δ-function excludes simultaneous poles in the two channels. This is the Veneziano model.

- In the case of the six-point amplitude one has six 2-particle channels and three 3-particles channels.
- One introduces six integration variables $X_{i,i+1} = u_{i,i+1}$ for the 2-particle channels, three variables $Y_{i,i+1,i+2} = u_{i,i+2}$ for the three 3-particle channels and six of the nine δ-functions to exclude simultaneous poles in incompatible channels.
- The nine δ-functions can be computed by drawing an hexagon.
- One gets for instance $\delta(u_{23} + u_{12}u_{35}u_{34} - 1)$ or $\delta(u_{24} + u_{12}u_{35}u_{45}u_{13} - 1)$ and so on.


1
2
3
4
5
6
p_1
p_2
p_3
p_4
p_5
p_6
u_{25}
u_{14}
u_{35}
u_{24}
u_{12}
u_{13}
u_{23}
u_{45}
u_{34}

- One then chooses the three independent variables left after the integration over the six δ-functions and writes the amplitude as follows:

$$\prod_{i=1}^{6}\int_0^1 X_i^{-\alpha(s_i)-1}\prod_{j=1}^{3}\int_0^1 Y_j^{-\alpha(s_j)-1}dX_1 dY_1 dX_5\prod_{i=1}^{6}\delta(C_i)dC_i$$

- Changing variables from the C_i to the remaining four X_i and two Y_j one should add a Jacobian in red getting

$$\prod_{i=1}^{6}\int_0^1 dX_i X_i^{-\alpha(s_{i,i+1})-1}\prod_{j=1}^{3}\int_0^1 dY_j Y_j^{-\alpha(s_{j,j+2})-1}\delta(X_2+X_1X_3Y_3-1)$$
$$\times\,\delta(X_3+X_2X_4Y_1-1)\delta(X_4+X_3X_5Y_2-1)\delta(X_6+X_1X_5Y_1-1)$$
$$\times\,\delta(Y_2+X_1X_4Y_1Y_3-1)\delta(Y_3+X_2X_5Y_1Y_2-1)\frac{1}{Y_2Y_3X_6^2}$$

- This is how the N-point was originally derived.
- No need of a Lagrangian !
  [Chan Hong-Mo and Tsou Sheung Tsun, Phys. Lett. **28B** (1969) 485, received Dec 6, 1968] See also
  [Goebel and Sakita, PRL **22** (1969) 257,received 16 Dec. 1968]
  [Bardakci and Ruegg, Phys. Lett. **28B** (1969) 671, rec. 22 Jan 1969,N=5]
  [Chan Hong-Mo, Phys. Lett. **28** B (1969) 425, rec. 16 Nov. 1978, up to N=6]
- The form of the amplitude that is mostly known is the so-called Koba-Nielsen amplitude

$$B_N = \int_{-\infty}^{+\infty} \frac{\prod_1^N dz_i \theta(z_i - z_{i+1})}{dV_{abc}} \prod_{i=1}^{N} \left[(z_i - z_{i+1})^{\alpha_0 - 1} \right] \prod_{j>i} (z_i - z_j)^{2\alpha' p_i \cdot p_j}$$

  with $\alpha_0 = -\alpha' m^2$ where m is the mass of the ext. scalar particles [Koba and Nielsen, 1969 (received 5 Feb 1969)]
- It was derived by using arguments similar to the previous ones and at the end it was brought in the previous most known form.

- What is the underlying structure that produced this amplitude?
- A simpler question is: what is the spectrum of particles contained in this N-point amplitude? What is their interaction?
- A particle corresponds to a simple pole in the scatt. amplitude.
- Our amplitude has in infinite number of poles.
- By factorising the amplitude at a certain pole determine the states that contribute to the residue $\Longrightarrow$ Spectrum of particles
  [Fubini and Veneziano,1969]
  [Bardakci and Mandelstam, 1969]
- This was the first attempt to extract the physical spectrum and see if it was free of ghosts (negative norm states that violate unitarity).

- In order to obtain the spectrum in a simpler way, the amplitude was first rewritten in terms of an infinite set of harmonic oscillators and a position and momentum operators that satisfy the following algebra

$$[a_{n\mu}, a^{\dagger}_{m\nu}] = \eta_{\mu\nu}\delta_{nm} \quad ; \quad [\hat{q}_\mu, \hat{p}_\nu] = i\eta_{\mu\nu}$$

  where $\eta_{\mu\nu} = (-1, 1, \dots 1)$ is the flat Minkowski metric in $D(4)$ dims [Fubini, Gordon and Veneziano, 1969; Fubini and Veneziano, 1969]. See also [Nambu, 1969] and [Susskind, 1969].
- Fubini and Veneziano introduced

$$Q^\mu(z) = \hat{q}^\mu - 2i\alpha' \hat{p}^\mu \log z + i\sqrt{2\alpha'} \sum_{n=1}^{\infty} \left(\frac{a_n^\mu}{\sqrt{n}} z^{-n} - \frac{a_n^{\dagger\,\mu}}{\sqrt{n}} z^n \right)$$

- and rewrote the amplitude as follows:

$$B_N = \int_{-\infty}^{\infty} \frac{\prod_1^N dz_i \theta(z_i - z_{i+1})}{dV_{abc}} \prod_{i=1}^{N} \left[(z_i - z_{i+1})^{\alpha_0 - 1} \right]$$

$$\times \langle 0 | \prod_{i=1}^{N} V(z_i, p_i) | 0 \rangle$$

where $\hat{p}_0|0\rangle = 0$, $a_n|0\rangle = 0$ and

$$V(z,p) =: e^{ipQ(z)} := e^{i\hat{q}k} e^{\sqrt{2\alpha'}k \sum_{n=1}^{\infty} \frac{a_n^\dagger}{\sqrt{n}} z^n} e^{-\sqrt{2\alpha'}k \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} z^{-n}} z^{2\alpha' \hat{p}k}$$

is the vertex operator corresponding to the external scalar particle.

- It satisfies the important properties:

$$\lim_{z \to 0} : e^{ipQ(z)} : |0\rangle = e^{ip\hat{q}}|0\rangle = |0, p\rangle \quad ; \quad \lim_{z \to \infty} \langle 0| z^2 : e^{ipQ(z)} := \langle 0, p|$$

- State-operator correspondence: later found in conformal theories [Belavin, Polyakov and Zamolodchikov, 1984].

- The generators of the projective group are

$$L_0 = \alpha' \hat{p}^2 + \sum_{n=1}^{\infty} n a_n^\dagger \cdot a_n \; ; \; L_1 = \sqrt{2\alpha'}\hat{p} \cdot a_1 + \sum_{n=1}^{\infty} \sqrt{n(n+1)} a_{n+1} \cdot a_n^\dagger$$

$$L_{-1} = L_1^\dagger = \sqrt{2\alpha'}\hat{p} \cdot a_1^\dagger + \sum_{n=1}^{\infty} \sqrt{n(n+1)} a_{n+1}^\dagger \cdot a_n$$

[F. Gliozzi; Chiu, Matsuda and Rebbi; Thorn, 1969]

- Transformation of the vertex operator under a projective transformation

$$[L_n, V(z,p)] = z^{n+1} \frac{dV(z,p)}{dz} + \alpha_0 (n+1) z^n V(z,p) \quad ; \quad n = 0, \pm 1$$

- The vertex operator is a conformal field with dimension α_0.
- By writing the amplitude in the previous way one discovers the presence of an underlying two-dimensional conformal field theory formalised only much later [Belavin, Polyakov and Zamolodchikov, 1984].

- Using the relation

$$z^{L_0} V(1,p) z^{-L_0} = V(z,p) z^{\alpha_0}$$

we can write the amplitude

$$A_{M+R} \equiv \langle 0, p_1 | V(1,p_2) D V(1,p_3) \dots D V(1, p_{N-1}) | 0, p_N \rangle = {}_M\langle p | D | p \rangle_R$$

- where $N = M + R$,

$$D = \int_0^1 dx x^{L_0 - 1 - \alpha_0} (1-x)^{\alpha_0 - 1} = \frac{\Gamma(L_0 - \alpha_0)\Gamma(\alpha_0)}{\Gamma(L_0)}$$

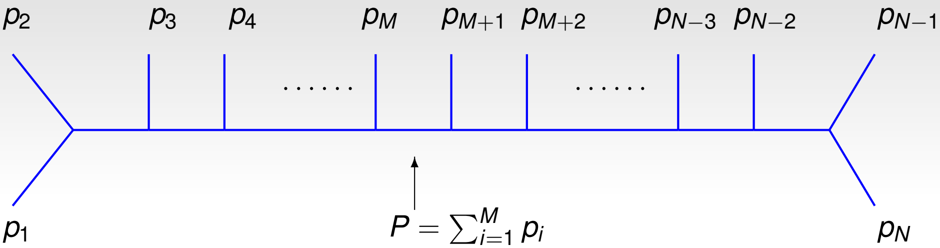
and

$${}_M\langle p| = \langle 0, p_1 | V(1,p_2) D V(1,p_3) \dots V(1, p_M)$$

and

$$|p\rangle_R = V(1, p_{M+1}) D \dots V(1, p_{M+R-1} | p_{M+R}, 0\rangle$$

- This form of the amplitude is very useful to study its factorisation properties at each pole.


p_2
p_3
p_4
p_M
p_{M+1}
p_{M+2}
p_{N-3}
p_{N-2}
p_{N-1}
p_1
$P = \sum_{i=1}^{M} p_i$
p_N

Factorisation properties

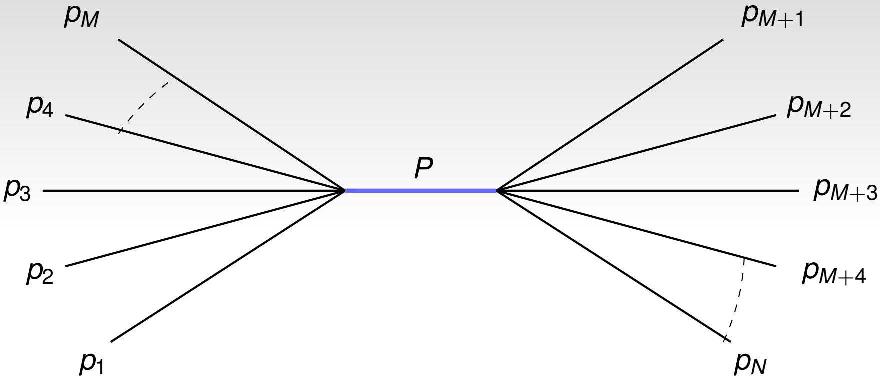
- They can be studied by inserting two completeness relations:

$$A_N = \sum_{\lambda,\mu} {}_M\langle p|\lambda, P\rangle\langle\lambda, P|D|\mu, P\rangle\langle\mu, P|p\rangle_R$$

$$= \sum_{\lambda,\mu} {}_M\langle p|\lambda, P\rangle \sum_{m=0}^{\infty}\langle\lambda, P|\frac{(-1)^m \left(\begin{array}{c}\alpha_0 - 1\\ m\end{array}\right)}{R + m - \alpha(-P^2)}|\mu, P\rangle|\langle\mu, P|p\rangle_R$$

where $R = \sum_{n=1}^{\infty} n a_n^\dagger a_n$ and the square of the exchanged momentum P is given by

$$P^2 = (p_1 + p_2 + \dots p_M)^2 = (p_{M+1} + p_{M+2} \dots + p_N)^2$$
$$\alpha(-P^2) = \alpha_0 - \alpha' P^2 \quad ; \quad -P^2 = M^2$$


p_M
p_4
p_3
p_2
p_1
P
p_{M+1}
p_{M+2}
p_{M+3}
p_{M+4}
p_N

- This expression shows that amplitude A_N has a pole in the channel $(1, M)$ when $\alpha(-P^2)$ is equal to an integer $n \geq 0$.
- The states $|\lambda\rangle$ that contribute to its residue are those satisfying the relation:

$$R|\lambda\rangle = (n - m)|\lambda\rangle \quad ; \quad m = 0, 1 \ldots n$$

- The number of independent states $|\lambda\rangle$ contributing to the residue gives the degeneracy of states for each level n.
- This provides the spectrum of particles contributing to the scattering amplitude.
- Manifest Lorentz invariance implies that the states contributing to the residues contain also states with negative norm.
- On the other hand, if those states are really coupled, they will violate unitarity.
- How can we have at the same time manifest Lorentz invariance and unitarity?

Decoupling conditions

- Look at QED to get inspiration.
- At the pole of a photon the amplitude is given by:

$$M_\mu(p_1, p_2 \ldots p_M; q)\frac{\eta^{\mu\nu}}{q^2}N_\nu(p_{M+1}, p_{M+2} \ldots p_{M+R}; q)$$

- The residue at the photon pole consists of four terms.
- One of them has a negative sign: it corresponds to a ghost.
- But we have to remember that gauge invariance implies

$$q_\mu M^\mu = q_\nu N^\nu = 0$$

- In the frame of reference where $q^\mu = (q, 0, 0, q)$ one is left with only two terms along the directions x and y corresponding to the two polarisations of a photon.
- Gauge invariance is necessary to have both manifest Lorentz invariance and absence of negative norm states (ghosts).
- Can something like that happen in the dual resonance model?

- Using the two relations

$$(L_1 - L_0)V(1,p) = V(1,p)(L_1 - L_0 + \alpha_0) \;\; ; \;\; L_1 x^{L_0} = x^{L_0+1} L_1$$

one can show that

$$(L_1 - L_0)V(1,p)D = V(1,p) \int_0^1 dx x^{L_0 - \alpha_0}(1-x)^{\alpha_0 - 1}(L_1 - L_0)$$

- Using them on the state $|p\rangle_R$ we see that $(L_1 - L_0)$ goes through all the products of a vertex and a propagator until we arrive at

$$(L_1 - L_0)|p\rangle_R = \ldots (L_1 - L_0)V(1, p_{R-1})|p_R, 0\rangle$$
$$= \ldots V(1, p_{R-1})(L_1 - L_0 + \alpha_0)|p_R, 0\rangle = 0$$

because $\alpha_0 - \alpha' p_R^2 = 0$

- In our case with an infinite number of oscillators one such condition is not enough to eliminate all ghosts.
- But additional conditions are present if $\alpha_0 = 1$ and therefore the lowest scalar particle is a tachyon with mass given by $\alpha' m^2 = -1$ [Virasoro, 1969]

- Virasoro introduced the new operators:

$$L_n = \sqrt{2\alpha' n}\hat{p} \cdot a_n + \sum_{m=1}^{\infty} \sqrt{m(n+m)} a_{n+m} \cdot a_m^\dagger$$
$$+ \frac{1}{2} \sum_{m=1}^{n-1} \sqrt{m(n-m)} a_{n-m} \cdot a_m \quad ; n \geq 0 \quad L_{-n} = L_n^\dagger$$

and the combination

$$W_n = L_n - L_0 - (n-1)$$

- Then using the relations:

$$W_n V(1;p) = V(1;p)(W_n + n) \quad ; \quad L_n x^{L_0} = x^{L_0+n} L_n$$
$$\Longrightarrow W_n V(1;p) D = V(1;p)[L_o + n - 1]^{-1} W_n \,,$$

where, for $\alpha_0 = 1$, he showed

$$W_n |p\rangle_R = 0 \quad ; \quad n = 1, 2 \ldots$$

- For $\alpha_0 = 1$ the propagator simplifies and is equal to $(L_0 - 1)^{-1}$, where $L_0 = \alpha' \hat{p}^2 + R$.
- The pole at $\alpha(-P^2) = n$ gets a contribution from the states satisfying the condition:

$$\sum_{m=1}^{\infty} m a_m^\dagger \cdot a_m = n \Longrightarrow (L_0 - 1)|\lambda, p\rangle = (R - n)|\lambda, p\rangle = 0$$

  We call them on-shell states at the level n.
- At the level $n = 0$ the only solution is $|0, p\rangle$ corresponding to the exchange of the tachyon with mass $\alpha' M^2 = -1$.
- At the level $n = 1$ we get a massless spin 1 particle $a_{1\mu}^\dagger|0, p\rangle$.
- At the level $n = 2$ we get

$$a_{1\mu}^\dagger a_{1\nu}^\dagger|0, p\rangle \quad ; \quad a_{2\mu}^\dagger|0, p\rangle \quad ; \quad \alpha' M^2 = 1$$

- At the level $n = 3$ we get

$$a^{\dagger}_{1\mu}a^{\dagger}_{1\nu}a^{\dagger}_{1\rho}|0,p\rangle \quad ; \quad a^{\dagger}_{2\mu}a^{\dagger}_{1\nu}|0,p\rangle \;\; ; \;\; a^{\dagger}_{3\mu}|0,p\rangle \quad ; \;\; \alpha' M^2 = 2$$

- The number of states is exponentially increasing with the mass.
- But, because of the decoupling conditions, how many of them really contribute?

Physical states conditions

- Consider the (off shell by m units) states $|\psi, P\rangle$ at the level $n-m$ satisfying the equations:

$$(L_0 + m - 1)|\psi, P\rangle = 0 \quad ; \quad 1 - \alpha' P^2 = n$$

$$R|\psi, P\rangle = \sum_{n=1}^{\infty} n a_n^\dagger \cdot a_n |\psi, P\rangle = (n-m)|\psi, P\rangle$$

- and from them, acting with L_{-m}, construct the states on shell at the level n:

$$L_{-m}|\psi, P\rangle \quad ; \quad (L_0 - 1)L_{-m}|\psi, P\rangle = 0 \;\; ; \;\; [L_{-m}, L_0] = -mL_{-m}$$

- We immediately see that they are decoupled from the physical states $|p_{(1,W)}\rangle$

$$\langle\psi, P|W_m|p\rangle_R = \langle\psi, P|(L_m - L_0 - m + 1)|p\rangle_R = 0$$

- The on-shell physical states are defined as those orthogonal to the previous states:

$$\langle \psi, P | L_m | Phys., P \rangle = 0 \Longrightarrow L_m | Phys., P \rangle = (L_0 - 1) | Phys., P \rangle = 0$$

$m = 1, 2 \ldots$ [Del Giudice and Di Vecchia, 1970]

- These equations do not completely define the physical subspace because there could be states that are physical (satisfying the previous equations), but that are decoupled from the states $|p\rangle_M$: they are zero norm states.

- The most general state at the level $N = 2$ is given by:

$$[\alpha^{\mu\nu} a^\dagger_{1,\mu} a^\dagger_{1,\nu} + \beta^\mu a^\dagger_{2,\mu}]|0, P\rangle$$

- In the center of mass frame where $P^\mu = (M, \vec{0})$ we get the following most general physical state ($\alpha' M^2 = 1$):

$$|Phys >= \alpha^{ij}[a^\dagger_{1,i} a^\dagger_{1,j} - \frac{1}{(D-1)}\delta_{ij} \sum_{k=1}^{D-1} a^\dagger_{1,k} a^\dagger_{1,k}]|0, P\rangle +$$

$$+\beta^i [a^\dagger_{2,i} - a^\dagger_{1,0} a^\dagger_{1,i}]|0, P\rangle +$$

$$+\alpha \left[\sum_{i=1}^{D-1} a^\dagger_{1,i} a^\dagger_{1,i} + \frac{D-1}{5}(a^{\dagger 2}_{1,0} - 2a^\dagger_{2,0}) \right] |0, P\rangle$$

where the indices i, j run over the $D-1$ space components.

- The first term corresponds to a spin 2 in D space-time dimensions and has a positive norm being made with space indices.

- The second term has zero norm, is orthogonal to the other physical states and it is decoupled from the states $|p\rangle_R$ since it can be written as

$$L_{-1} a_{1,i}^{+} |0, P\rangle$$

- The last state is spinless and has a norm given by:

$$2(d-1)(26-d)$$

- If $d < 26$ it corresponds to a physical spin zero particle with positive norm.
- If $d > 26$ it is a ghost.
- If $d = 26$ it has zero norm, is orthogonal to the other physical states and it is decoupled from the states $|p\rangle_R$ since it can be written as:

$$(2L_2^\dagger + 3L_1^{\dagger 2})|0, P > \quad ; \quad 1 - \alpha' P^2 = 2$$

- The same analysis can be done at any massive level, but it becomes more and more complicated.

Vertex operators for excited states

- The previous analysis was done starting from the N-tachyon amplitude.
- It could have been done starting from an amplitude involving any physical state.
- We can associate to any physical state $|\alpha, p\rangle$ its corresponding vertex operator $V_\alpha(z, p)$ that is a conformal field with dimension $\Delta = 1$:

$$[L_n, V_\alpha(z, p)] = \frac{d}{dz}\left(z^{n+1} V_\alpha(z, p)\right)$$

- It reproduces the physical state in the limits:

$$\lim_{z\to 0} V_\alpha(z, p)|0, 0\rangle \equiv |\alpha; p\rangle \;\; ; \;\; \langle 0; 0| \lim_{z\to\infty} z^2 V_\alpha(z, p) = \langle \alpha, -p|$$
$$L_n|\alpha, p\rangle = (L_0 - 1)|\alpha, p\rangle = 0 \;\; ; \;\; n = 1, 2 \ldots$$

- It satisfies the hermiticity relation:

$$V_\alpha^\dagger(z,p) = V_\alpha(\frac{1}{z}, -p)(-1)^m \ ; \ 1-\alpha' p^2 = n$$

[Campagna, Fubini, Napolitano and Sciuto, 1970]
[Clavelli and Ramond, 1970]

- In terms of these vertices one can write the most general amplitude involving physical states:

$$B_N(\alpha_1, p_1; \ldots \alpha_N, p_N)$$
$$= \int_{-\infty}^{\infty} \frac{\prod_1^N dz_i \theta(z_i - z_{i+1})}{dV_{abc}} \langle 0,0| \prod_{i=1}^{N} V_{\alpha_i}(z_i, p_i)|0,0\rangle$$

- It has precisely the same form as the N-tachyon amplitude except that the vertex operators depend on the physical states involved.
- There is a complete democracy among the physical states, as advocated by the followers of S-matrix theory [G. F. Chew, The analytic S matrix, A basis for nuclear democracy, 1966].

- Chew writes at the end of his book:
  Perhaps a lucky guess about some new symmetry group- with an associated set of fundamental particles-is destined to lead us quickly out of the wilderness.
  These are perhaps the chief reasons for the current attention to quarks, and if truly elementary particles are discovered, the analytic S matrix as a fundamental approach to strong interactions will lose much of its appeal. The accumulated weight of experimental evidence over a span of more than 30 years, however, makes the existence of hadron aristocrats seem unlikely. Physicists, whether they like it or not, probably will have to find a strong interaction theory compatible with nuclear democracy.
- This did not happen for the strong interactions because we know today that they are described by QCD that is a field theory involving quarks and gluons and is a non-abelian generalisation of QED.
- Luckily QCD was found after the DRM. Otherwise may be string theory would not have been found [J. Schwarz].

- On the other hand, the previous ideas based on S-matrix theory strongly contributed to the discovery of the Veneziano model and are also fashionable today to get constraints on how a consistent theory should look like.
- and may be constrain so much the theory that one may end up with a unique solution.
- Now we know that we are dealing with a two-dimensional conformal field theory because the operators L_n satisfy the Virasoro algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{D}{12} n(n^2-1)\delta_{n+m;0} \quad ; \quad L_n = z^{-n}\frac{d}{dz}$$

  [Fubini and Veneziano, 1969; J. Weis, 1970].
- The vertex operator associated to the massless vector state is somewhat special and will play an important role in the proof of the no-ghost theorem.
- It is given by:

$$V_\epsilon(z,k) \equiv \epsilon \cdot \frac{dQ(z)}{dz} e^{ik\cdot Q(z)} \quad ; \quad k\cdot\epsilon = k^2 = 0$$

DDF operators

- We want to construct an infinite set of physical states using the vertex operator of the massless spin 1 state.
- The starting point is the DDF operator defined in terms of the vertex operator corresponding to the massless gauge field:

$$A_{i,n} = \frac{i}{\sqrt{2\alpha'}} \oint_0 \frac{dz}{2\pi i} \epsilon_i^\mu P_\mu(z) e^{ik\cdot Q(z)}$$

where

$$P(z) \equiv \frac{dQ(z)}{dz} = -i\sqrt{2\alpha'} \left[\sqrt{2\alpha'} \frac{\hat{p}_0}{z} + \sum_{n=1}^{\infty} \sqrt{n} \left(a_n z^{n-1} + a_n^\dagger z^{-n-1} \right) \right]$$

- The index i runs over the $D-2$ transverse directions that are orthogonal to the momentum k.
[Del Giudice, Di Vecchia and Fubini, 1971]

- The DDF operators commutes with the gauge operators L_m:

$$[L_m, A_{n;i}] = 0$$

because the vertex operator transforms as a total derivative under the action of L_n.

- They satisfy the algebra of the harmonic oscillator:

$$[A_{n,i}, A_{m,j}] = n\delta_{ij}\delta_{n+m;0} \;\; ; \;\; i,j = 1 \ldots D-2$$

- They span a positive definite infinite dimensional space.
- Unfortunately not complete in $D = 4$.

The no-ghost theorem

- Going back to level $n = 2$ we have the following DDF states contributing at this level:

$$A_{-1,i}A_{-1,j}|0,p\rangle \quad ; \quad A_{-2,i}|0,p\rangle \;\; ; \;\; i,j = 1 \ldots D-2$$

- Therefore the number of states contributing is equal to

$$\frac{(D-2)(D-1)}{2} + D - 2 = \frac{(D-2)(D+1)}{2} \tag{1}$$

- that is equal to the number of components of the state:

$$[a^{\dagger}_{1,I}a^{\dagger}_{1,J} - \frac{1}{(D-1)}\delta_{IJ}\sum_{K=1}^{D-1} a^{\dagger}_{1,K}a^{\dagger}_{1,K}]|0,P\rangle \quad ; \quad I,J = 1 \ldots D-1$$

  given by:

$$\frac{(D-1)D}{2} - 1 = \frac{(D-2)(D+1)}{2}$$

  describing a spin 2 in $D-1$ space dimensions.
- This state is the only physical state at the level $n = 2$ if $D = 26$.

- For $D = 26$ the DDF states are a complete set of states at the level $n = 2$.
- It turns out, after a detailed analysis, that, if $D = 26$, they are indeed a complete set of states at an arbitrary level n. [Goddard and Thorn, 1972 and Brower, 1972]
- Since they span a positive definite Hilbert space, this means that the DRM with $\alpha_0 = 1$ is ghost-free if $D = 26$.
- It can be shown that this is also true for any $D < 26$.
- However, in this case there are additional operators to be included besides the DDF ones.
- The states produced by these additional operators are called Brower states [Brower, 1972].
- They are needed already at the level $n = 2$ to take care of the additional scalar state not taken into account by the DDF states.

Field theory Limit

- The DRM is not in contradiction with field theory, but is an extension of field theory.
- By performing the zero slope limit ($\alpha' \to 0$) one gets gauge theories and gravity [Neveu and Scherk, 1972; Yoneya, 1973]
- But we can do the zero slope limit in many different ways and select many consistent field theory [Scherk, 1971; Di Vecchia, Magnea, Lerda, Marotta and Russo, 1996; Frizzo, Magnea and Russo, 2000; Magnea, Playle, Russo and Sciuto, 2015].
- However it does not seem possible to construct a consistent string extension of a theory that is not a gauge theory or gravity.

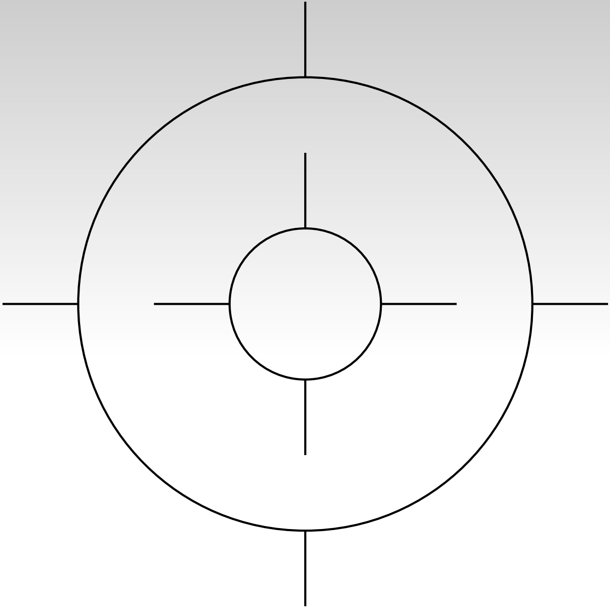
$D = 26$ from the non-planar loop

- Historically, the critical dimension was not found as described before.
- It was first found in the study of one-loop amplitudes.
- The Veneziano model and its extension, the N-point function, satisfies all the axioms of S matrix theory except unitarity.
- In fact, unitarity in a model with only resonances imposes that the total width of a resonance Γ must be the sum of the partial widths over all the possible decay channels:

$$\Gamma = \sum_i \Gamma_i$$

- If the model is ghost-free, all partial widths are positive definite and a sum of positive numbers cannot give zero unless $\Gamma_i = 0$ for any i.
- In the Veneziano model, the total width $\Gamma = 0$, but the partial widths are non zero $\Longrightarrow$ unitarity is violated !

- Immediately after the discovery of the Veneziano model, it was proposed to make it unitary by adding to it the contribution of loop diagrams.
- By doing so, one generates the branch points required by unitarity.
- At one-loop level in the DRM, two kinds of loop diagrams appear: the planar and the non-planar.
- They correctly generate the branch cuts required by unitarity, but the non-planar one showed additional branch cuts violating unitarity.



Non-planar diagram with 4 part. on circle 1 and 4 part. on circle 2 (8-point amplitude). In the planar diagram 8 part. on the same circle.

- In 1970, Lovelace noticed that these branch cuts become poles if the dimension of the space-time $D = 26$.
- And poles create no problem with unitarity.
- They are just additional states appearing at one-loop level.
- Today we know that, while the original poles correspond to the excitation of an open string, the new poles correspond instead to the excitation of a closed string.
- They both lie on linear Regge trajectories given respectively, by:

$$\alpha_{open}(s) = 1 + \alpha' s \quad ; \quad \alpha_{closed}(s) = 2 + \frac{\alpha'}{2} s$$

- At that time, practically nobody took Lovelace's observation seriously.
- But this has been the first evidence of the existence of a critical dimension.

- In parallel with the Veneziano model and its extension to N external particles, another four-particle amplitude and its extension to N external legs was discussed.
- It goes under the name of Shapiro-Virasoro model.
- Its four-point amplitude is

$$A_4(s,t,u) \sim \frac{\Gamma(-\frac{\alpha(s)}{2})\Gamma(-\frac{\alpha(t)}{2})\Gamma(-\frac{\alpha(u)}{2})}{\Gamma(1+\frac{\alpha(s)}{2})\Gamma(1+\frac{\alpha(t)}{2})\Gamma(1+\frac{\alpha(u)}{2})}$$

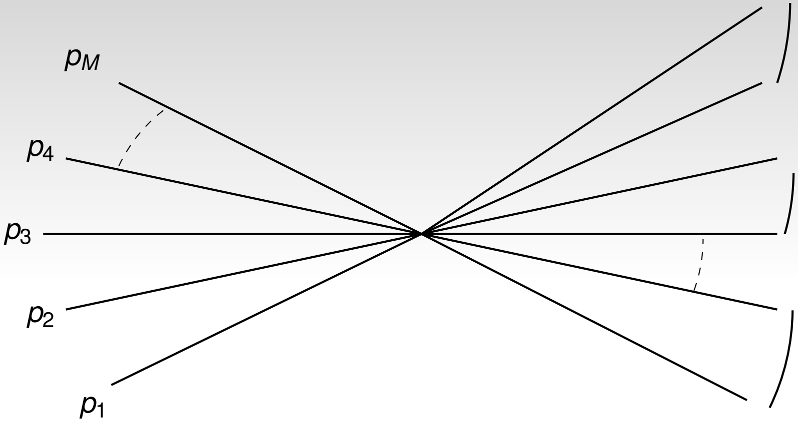
- Its N point amplitude is given by:

$$A_N = \int \frac{\prod_{i=1}^{N} d^2 z_i}{dV_{abc}} \prod_{i<j} |z_i - z_j|^{\alpha' p_i p_j} \;\; ; \;\; 2 + \frac{\alpha'}{2} m^2 = 0$$

where now three complex variables z_a, z_b, z_c can be fixed.
- While open string contains gauge theories, closed string theory contains gravity.
- These are the interactions that we observe in nature.

Multiloops

- Multiloop amplitudes were also already computed in 1969, even before it was clear that the DRM was related to string theory. [Lovelace, Phys. Lett. B **32** (1970) 490- 494 and 703-708; Alessandrini, Amati, Le Bellac and Olive, Phys. Rep. **1** (1971) 269-346.]
- At multiloop level the measure of integration over the moduli was not yet known at that time.
- It was computed only after BRST invariance was formulated for string theory.



BRST invariant (M+2R)-reggeon vertex saturated with R BRST invariant propagators

- The h loop amplitude with the emission of N tachyons is given by:

$$A_N^{(h)} \sim \int [dm]_M^h \prod_{i<j} \left[\frac{\exp\left(\mathcal{G}^{(h)}(z_i, z_j)\right)}{\sqrt{V_i'(0) V_j'(0)}} \right]^{2\alpha' p_i p_j}$$

[Di Vecchia, Frau, Lerda and Sciuto; Petersen and Sidenius, 1987]

- $\mathcal{G}(z_i, z_j)$ is the h-loop Green function:

$$\mathcal{G}^{(h)}(z_i, z_j) = \log E^{(h)}(z_i, z_j) - \frac{1}{2} \int_{z_i}^{z_j} \omega^\mu (2\pi Im\tau_{\mu\nu})^{-1} \int_{z_i}^{z_j} \omega^\nu$$

- The integration measure is given by:

$$[dm]^h_M = \frac{1}{dV_{abc}} \prod_{i=1}^{N} \frac{dz_i}{V'_i(0)} \prod_{\mu=1}^{h} \left[\frac{dk_\mu d\xi_\mu d\eta_\mu}{k^2_\mu (\xi_\mu - \eta_\mu)^2} (1 - k_\mu)^2 \right]$$
$$\times [\det(-i\tau_{\mu\nu})]^{-D/2} \prod_\alpha{}' \left[\prod_{n=1}^{\infty} (1 - k^n_\alpha)^{-D} \prod_{n=2}^{\infty} (1 - k^n_\alpha)^2 \right]$$

in terms of the loop variables k_μ, η_μ, ξ_μ in the Schottky parametrisation of the loop diagrams.

- $V_i(z)$ describes the parametrisation around the puncture and is such that $V_i(0) = z_i$.

Towards string theory

- Given the infinite number of harmonic oscillators it was formulated already in 1969 the idea that the underlying theory was the theory of a relativistic string [Nambu, Nielsen and Susskind, 1969].
- The Lagrangian of a relativistic string, generalising that of a point particle, was written in 1970, but at that time one did not know how to treat it [Nambu, Goto, 1970].
- We have to remember that the Faddeev-Popov ghost was introduced for non-abelian gauge theories in those years and the BRST quantisation of gauge theories appeared around 1974.
- In other words people were starting to understand how to quantise theories having redundant degrees of freedom as gauge theories, gravity and also string theory.
- Another problem of discussion in those years was that, in the DRM the Virasoro algebra was a gauge algebra, while in the free scalar theory in two dimensions was a classification algebra as in conformal field theory.

- In the case of a point particle the action is proportional to the length of the path:

$$S_{particle} = -mc \int \sqrt{-dx^\mu dx_\mu} = -mc \int d\tau \sqrt{-\dot{x}^2}$$

while, in the case of a string, is proportional to the area of the surface spanned by the string

$$S_{string} = -\frac{T}{c} \int \sqrt{-d\sigma_{\mu\nu} d\sigma^{\mu\nu}} = -\frac{T}{c} \int d\sigma d\tau \sqrt{(\dot{x}x')^2 - \dot{x}^2 (x')^2}$$

where $\dot{x} = \frac{\partial x}{\partial \tau}$ and $x' = \frac{\partial x}{\partial \sigma}$.

- It can be rewritten in terms of the worldsheet metric $g_{\alpha\beta}$:

$$S = -\frac{T}{2} \int d^2\sigma \sqrt{-g} g^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu}$$

- A partial choice of gauge is the conformal gauge where

$$g_{\alpha\beta} = \rho \eta_{\alpha\beta}$$

- In this gauge we are left with a linearised Lagrangian and a constraint:

$$S = -\frac{T}{2} \int d^2\sigma \eta^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu}$$

$$T_{\alpha\beta} = \partial_\alpha x^\mu \partial_\beta x_\nu - \frac{1}{2}\eta_{\alpha\beta}\partial_\gamma x^\mu \partial^\gamma x_\mu = 0$$

- We have to wait until the end of 1972 to have the proof that the spectrum extracted from the string Lagrangian, quantised in the light-cone gauge, was the same as that extracted from the DRM with $\alpha_0 = 1$ [Goddard, Goldstone, Rebbi and Thorn, 1973].
- They fix completely the gauge going into the light-cone gauge

$$x^+ = \frac{1}{\sqrt{2}}\left(x^0 - x^{D-1}\right) = 2\alpha' p^+ \tau$$

- and then use the classical conditions $L_n = 0$ to write

$$\dot{x}^- = \frac{1}{4\alpha' p^+} \sum_{i=1}^{d-2} \left(\dot{x}_i^2 + {x'_i}^2\right) \quad ; \quad x'^- = \frac{1}{2\alpha' p^+} \sum_{i=1}^{d-2} \dot{x}_i x'_i$$

- They also constructed the Lorentz generators only in terms of the transverse degrees of freedom and showed that the Lorentz algebra was correctly reproduced only if $D = 26$.
- The quantisation of an open string reproduced the spectrum of the DRM, while that of a closed string reproduced that of the Shapiro-Virasoro model.
- The computation of the zero point energy of the string gave the intercept $\alpha_0 = 1$ for $D = 26$ [Brink and Nielsen, 1973]

$$\alpha_0 = -\frac{D-2}{2}\sum_{n=1}^{\infty} n = -\frac{D-2}{2}\lim_{s\to -1}\sum_{n=1}^{\infty} n^{-s} = -\frac{D-2}{2}\lim_{s\to -1}\zeta(s) = 1$$

for $D = 26$ and $\zeta(-1) = -\frac{1}{12}$ [Gliozzi, 1976].

- Immediately after it was shown that the three-point coupling of three strings [Cremmer and Gervais, 1973; Mandelstam, 1973] reproduced the coupling of three arbitrary DDF states computed by [Ademollo, Del Giudice, Di Vecchia and Fubini, 1974].
- The N-point amplitude has then been computed adding to the free string action an interaction term with an external field by [Ademollo et al, 1974] or by using the path integral in the light-cone gauge [Mandelstam, 1974].
- It turned out that the only allowed external fields are those corresponding to a state of the string.
- Originally one wanted to construct the S-matrix describing the scattering of strong interacting particles (mesons).
- By requiring consistency one ended up in constructing the S-matrix for a theory unifying gauge theories with gravity.

What about the pions?

- The Lovelace(1968)-Shapiro(1969) model

$$A^{(\mathrm{LS})}[1,2,3,4] = -\frac{1}{2\pi\alpha' F_\pi^2} \frac{\Gamma(1-\alpha_\rho(s))\Gamma(1-\alpha_\rho(t))}{\Gamma(1-\alpha_\rho(s)-\alpha_\rho(t))}$$

$\alpha_\rho(t) = \frac{1}{2} - \alpha' m_\pi^2 + \alpha' t$ and $\alpha' = \frac{1}{2(m_\rho^2 - m_\pi^2)}$

- For $\alpha' \to 0$ it reduces to the 4-pion amplitude of the non-linear σ model that describes the low-energy behaviour of QCD.
- It has critical dimension $D = 4$ and probably no ghosts.
- But any proposed extension to N pions contains negative norm states that violate unitarity.
- The only known consistent extension to N legs is the one where $\alpha_0 = 1$ corresponding to the amplitude of the Neveu-Schwarz model where the pion becomes a tachyon.
- This particle together with all the states lying on Regge trajectories with half integer intercepts are then eliminated by the GSO projection.

- In this way one obtains the bosonic spectrum of type I superstring theory.
- No consistent string extension of the non-linear σ model exists.
- The only theories that admit a consistent string extension are gauge theories and gravity.

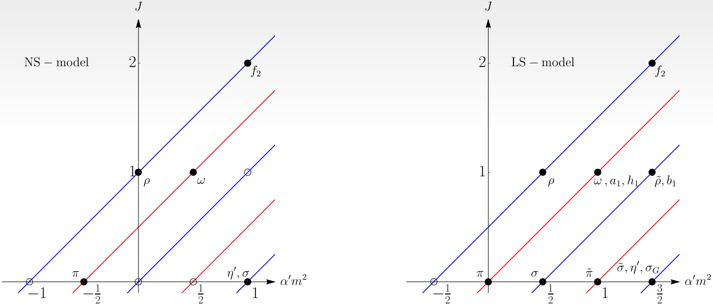


Figure: Spectrum of NS (left) and LS (right) model in four dimensions, Regge trajectories in blue (red) have G-parity $+1$ (-1). Bullets represent 'physical' states, open circles represent 'missing' states.

Conclusions

- String theory is peculiar in the sense that the S-matrix and the spectrum have been obtained before knowing the Lagrangian.
- The Lagrangian came later and was not necessary to compute the observable quantities.
- It turns out that the S-matrix obtained unifies a string extension of the gauge theories with gravity.
- They always come together.
- It does not exist a string theory with one of them without the other.
- To my knowledge there is no consistent string extension of other theories.
- String extensions of the non-linear σ-model were constructed by [J.J. Carrasco, C. Mafra and O. Schlotterer (2016)] [C. Mafra and O. Schlotterer (2016)] [N. Arkani-Hamed, Qu Cao, Jin Dong, C. Figueiredo and Song He (2023)], [Bianchi, Consoli and Di Vecchia, 2021] [Jin Dong, Xiang Li, and Fan Zu (2024)].
- All of them have ghosts.

- We are still waiting to get a string theory for hadrons.
- Probably this is not possible with linearly rising Regge trajectories and with an infinitely thin string.
- On the other hand, being a theory unifying gauge theory with gravity, one could have expected that string theory could provide an understanding of the Standard Model and of the physics beyond it.
- In fact in $D = 11$ we have a unique theory (M theory) or in $D = 10$ we have few string theories all related to each other.
- But we live in $D = 4$ and in going from 10 to 4 one gets so many possibilities that one loses in predictivity.

THANK YOU FOR YOUR ATTENTION

ADDITIONAL MATERIAL

Conclusions

- Open string theories include gauge theories, while closed string theories include gravity.
- If one starts from an open string theory, one gets closed strings at one loop level through the non-planar loop.
- If one starts from a closed string theory one gets non-perturbative objects, called Dp-branes, that have the property of having open strings attached to their world-volume.
- In conclusion, one cannot have a gauge theory without also having gravity and viceversa.
- That is also what we observe in nature.
- In $D = 10$ one gets few string theories (2 heterotics, type IIA and IIB and type I) having only one free parameter: the string length $\ell_s = \sqrt{\alpha' \hbar}$ (to be compared with Planck length $= \sqrt{G_N \hbar}$) plus of course c and $\hbar$.

- String theory is not in contradiction with field theory, but is an extension of field theory.
- By performing the zero slope limit ($\alpha' \to 0$) one gets gauge theories and gravity [Neveu and Scherk, 1972; Yoneya, 1973]
- But we can do the zero slope limit in many different ways and select practically any consistent field theory that we want [Scherk, 1971; Di Vecchia, Magnea, Lerda, Marotta and Russo, 1996; Frizzo, Magnea and Russo, 2000; Magnea, Playle, Russo and Sciuto, 2015].
- String theory has the characteristic of being a theory of everything, not only because it describes all interactions observed in nature, but especially because it can interact only with external fields corresponding to one of the string states.
- String theories are more or less unique in $D = 10$, but we live in $D = 4$!

- We need to compactify six of the ten dimensions and this can be done in many different ways.
- We cannot use string theory to predict or to explain what we observe at accelerator's energy. But the same is also true in FT.
- We can only search compactifications that contain the Standard Model, classify them, check if they are stable, stabilise the moduli and write down the four-dimensional effective Lagrangian.
- It is not an easy task.

Zero-norm spurious and physical states

- A set of them at the level n can be generated as follows.
- Let us consider a physical state $|\psi_1, P\rangle$ at the level $n-1$ that is off shell by one unit:

$$L_0|\psi_1, P\rangle = L_n|\psi_1, P\rangle = 0 \;\; ; \;\; n = 1, 2 \dots \;\; ; \;\; 1 - \alpha' P^2 = n$$

- Starting from any of the previous states we can construct an on shell physical state at the level n as follows:

$$L_{-1}|\psi_1, P\rangle \Longrightarrow L_n(L_{-1}|\psi_1, P\rangle) = (L_0 - 1)(L_{-1}|\psi_1, P\rangle) = 0 \qquad (2)$$

- that is decoupled from the states $|p\rangle_R$:

$$\langle\psi_1, P|(L_1 - L_0)|p\rangle_R = 0 \qquad (3)$$

- All those states have zero norm:

$$\langle\psi_1, P|L_1 L_{-1}|\psi_1, P\rangle = \langle\psi_1, P|\,(2L_0 + L_{-1}L_1)\,|\psi_1, P\rangle = 0$$

- It can be shown that Eqs. (2) and (3) can be satisfied only by zero norm states.

More on DDF states

- The zero mode part of $Q(z) = \cdots - 2\alpha' i\hat{p} \log z \ldots$ has a logarithmic singularity at $z = 0$.
- The contour integral is well defined only if we constrain the momentum of the state, on which $A_{i,n}$ acts, to satisfy the relation:

$$2\alpha' p \cdot k = n$$

where n is a non-vanishing integer.
- We get

$$[A_{n,i}, A_{m,j}] = -\frac{1}{2\alpha'} \oint_0 \frac{d\zeta}{2\pi i} \oint_\zeta \frac{dz}{2\pi i} \epsilon_i \cdot P(z) e^{ik\cdot Q(z)} \epsilon_j \cdot P(\zeta) e^{ik'\cdot Q(\zeta)}$$

where

$$2\alpha' p \cdot k = n \;\; ; \;\; 2\alpha' p \cdot k' = m$$

- k and k' are supposed to be in the same direction, namely

$$k_\mu = n\hat{k}_\mu \quad ; \quad k'_\mu = m\hat{k}_\mu$$

with

$$2\alpha' p \cdot \hat{k} = 1$$

- Since $\hat{k} \cdot \epsilon_i = \hat{k} \cdot \epsilon_j = \hat{k}^2 = 0$ a singularity for $z = \zeta$ can appear only from the contraction of the two terms $P(\zeta)$ and $P(z)$ that is given by:

$$\langle 0, 0|\epsilon_i \cdot P(z)\epsilon_j \cdot P(\zeta)|0, 0\rangle = -\frac{2\alpha'\delta_{ij}}{(z-\zeta)^2}$$

- From it we get:

$$[A_{n,i}, A_{m,j}] = \delta_{ij} in \oint_0 \frac{d\zeta}{2\pi i} \hat{k} \cdot P(\zeta) e^{i(n+m))\hat{k}\cdot Q(\zeta)} =$$

$$= in\delta_{ij}\delta_{n+m;0} \oint_0 \frac{d\zeta}{2\pi i} \hat{k} \cdot P(\zeta) \;\; ; \;\; P(\zeta) = -2i\alpha' \frac{\hat{p}}{\zeta} + \ldots$$

- We have used the fact that the integrand is a total derivative and therefore one gets a vanishing contribution unless $n + m = 0$.
- We get:

$$[A_{n,i}, A_{m,j}] = n\delta_{ij}\delta_{n+m;0} \;\; ; \;\; i, j = 1 \ldots D-2$$

- In terms of this infinite set of transverse oscillators we can construct an orthonormal set of states:

$$|i_1, N_1; i_2, N_2; \dots i_m, N_m\rangle = \prod_h \frac{1}{\sqrt{\lambda_h!}} \prod_{k=1}^{m} \frac{A_{i_k, -N_k}}{\sqrt{N_k}} |0, p\rangle$$

where λ_h is the multiplicity of the operator $A_{i_h, -N_h}$ in the product.

- They all have positive definite norm and satisfy the on shell physical conditions:

$$(L_0 - 1)|i_1, N_1; i_2, N_2; \dots i_m, N_m\rangle = L_n|i_1, N_1; i_2, N_2; \dots i_m, N_m\rangle = 0$$

for $n = 1, 2 \dots$, because the DDF oscillators commute with any Virasoro operator and the tachyon state $|0, p\rangle$ satisfies the previous conditions .

- The momentum of the state and its mass are given by

$$P = p - \sum_{i=1}^{m} \hat{k} N_i \;\; ; \;\; 1 - \alpha' P^2 = \sum_k N_k = n$$

Superconvergence relations and FESR

- We introduce the variable ν

$$\nu = \frac{s-u}{4}$$

to treat in a symmetric way the s and u channels.

- The scattering amplitude $T(\nu)$ has only singularities on the real axis corresponding to poles and the various two-particle cuts, three-particle cuts and so on.
- We can use the Cauchy theorem to write

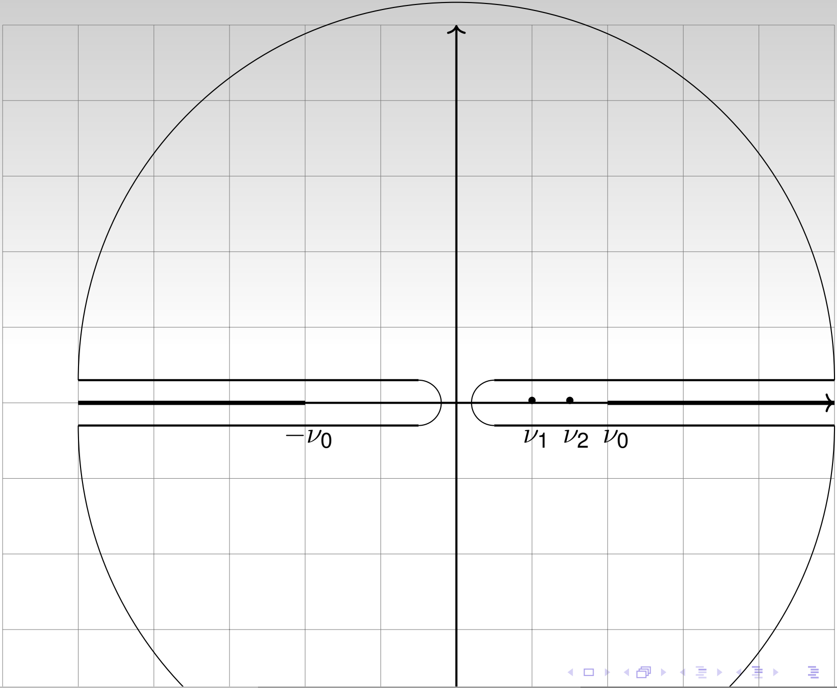
$$T(\nu) = \frac{1}{2\pi i} \oint_C d\nu' \frac{T(\nu')}{\nu - \nu'}$$

- If $T(\nu) \to |\nu|^{-N-1}$ for positive N ($\nu \to \infty$) then we obtain a set of superconvergence relations:

$$\int_{-\infty}^{+\infty} \nu^n ImT(\nu) d\nu = 0 \;\; ; \quad n = 0, \dots N$$

- We have used the relation:

$$\lim_{\epsilon\to 0}\Big(T(s+i\epsilon)-T(s-i\epsilon)\Big)=\lim_{\epsilon\to 0}\Big(T(s+i\epsilon)-T^*(s+i\epsilon)\Big)=2i\mathrm{Im}T(s)$$


$-\nu_0$
ν_1
ν_2
ν_0

Partial and total widths

- In the case of spin zero resonance in the s-channel with mass M_R and total width Γ the scattering amplitude has a pole at $s = s_R + i\Gamma M_R$ ($_R = m_R^2$) and is given by:

$$T_{fi} = \frac{2M_R A_{fi}}{s_R - s - i\Gamma M_R} \;\; ; \;\; T^{\dagger}_{fi} = \frac{2M_R A_{fi}}{s_R - s + i\Gamma M_R}$$

- Then the l.h.s. of the unitarity relation is given by

$$i(T_{fi} - T^{*}_{fi}) = -\frac{(2M_R)^2 \Gamma A_{fi}}{(s - s_R)^2 + \Gamma^2 M_R^2}$$

- We can similarly introduce the scattering amplitudes:

$$T_{fn} = \frac{2M_R A_{fn}}{s_R - s - i\Gamma M_R} \;\; ; \;\; T^{\dagger}_{ni} = \frac{2M_R A_{ni}}{s_R - s + i\Gamma M_R}$$

- Use them to compute

$$(2\pi)^4 \sum_n \delta(p_f - p_n) T_{fn} T^{\dagger}_{ni} = \sum_n \frac{(2M_R)^2 A_{fn} A_{ni}}{(s - s_R)^2 + M_R \Gamma^2}$$

- Comparing the two we get

$$\Gamma A_{fi} = \sum_n A_{fn} A_{ni}$$

- By writing $A_{fi} = A_f A_i$ we get

$$\Gamma = \sum_n A_n^2 \Longrightarrow \Gamma = \sum_n \Gamma_n \quad , if \quad \Gamma_n = A_n^2$$